JOURNAL OF COMBINATORIAL THEORY, Series A 26, 90–95 (1979)

Note

Locally Polar Lattices*

WILLIAM M. KANTOR

Department of Mathematics, University of Oregon, Eugene Oregon 97403

Communicated by the Managing Editors

Received June 18, 1977

Lattices are studied and characterized in which all intervals above points are polar spaces.

A lattice $\mathscr{L}$ is *locally polar* if each element is a join of points (atoms) and each interval $\mathscr{L}^x = \{y \mid x \leqslant y\}$, x a point, is a polar space (see Tits [6] or Buekenhout and Shult [4]). Recall that the *rank* of a polar space is the maximum projective dimension of an element; since this number is finite, $\mathscr{L}$ has a 1 and a 0. There is an obvious dimension function on $\mathscr{L}$, so lines and planes have the obvious meaning. We assume that $\mathscr{L}$ has the following properties.

(i) Each $\mathscr{L}^x$ has rank $n \geqslant 3$; if x and x' are distinct points, then $x \vee x'$ is 1 or a line.

(ii) If D, E, F, are planes such that $D \wedge E$ and $E \wedge F$ are lines, then there is a point x such that $x \vee D$, $x \vee E$ and $x \vee F$ are 3-spaces.

(iii) (Connectedness.) Given points p and q, there exist points $p = x_0, x_1, \ldots, x_k = q$ such that $x_{i-1} \vee x_i$ is a line for $i = 1, \ldots, k$.

(iv) Three pairwise collinear points are always coplanar.

THEOREM. *If $\mathscr{L}$ is a locally polar lattice satisfying* (i) *through* (iv), *then there is a canonical embedding of $\mathscr{L}$ into a polar space of rank $n + 1$.*

In particular, $\mathscr{L}$ can be canonically embedded in a projective space of dimension at most $2n + 3$, by the deep result of Tits [6].

This theorem has an obvious application to the program in Buekenhout [2]. Related results are found in Buekenhout [1] and Buekenhout and Hubaut [3]. Note, however, that our $\mathscr{L}$ need not be finite and that lines may have more than two points.

The proof of the theorem is a straightforward application of the method in [5]: we introduce ideal points and lines in a fairly natural way, and then

* This research was supported in part by NSF Grant MCS 76-07268.

0097-3165/79/010090–06$02.00/0

appeal to the Buekenhout–Shult theorem [4]. After the proof, several examples are presented.

Notation. For $X \in \mathscr{L}$, dim X is one less than the minimum number of points with join X. For X, $Y \in \mathscr{L}$ we write $X \sim Y$ when $\dim X \vee Y = 1 + \max\{\dim X, \dim Y\}$. Thus, (ii) states that $x \sim D, E, F$.

Points are denoted by p, x, y, z, lines by K, L, M, N, X, Y, Z, and planes by D, E, F.

Abbreviate $\{L, M\}$ by LM whenever $L \sim M$.

Define $LM \circ LN$ whenever there is an $x \sim L \vee M, L \vee N$ with $(x \vee L) \wedge (x \vee M) = (x \vee L) \wedge (x \vee N)$; the latter element is then a line. Let $\equiv$ denote the equivalence relation generated by $\circ$ on the set of symbols LM.

The equivalence classes of $\equiv$ are called *ideal points* and denoted $\alpha, \beta, \gamma, \delta$. (Ideal lines are defined later.) If $LM \in \alpha$ we write $\alpha < L$ or $L > \alpha$. The most difficult part of the proof of the theorem is the following fact.

LEMMA 1. *Assume* (i) *and* (ii), *and let* $KL \in \alpha$.

(a) *If* $K \wedge L = p$, *then* α *consists of all pairs of coplanar lines on* p.
(b) *If* $K \wedge L = 0$, *then each point is on at most one line* $>\alpha$.

The proof of this lemma, and of the theorem itself, will be given in a sequence of steps.

(I) If $1 \neq S \in \mathscr{L}$ and $\dim S \geqslant 4$, then S is canonically embeddable in a projective space. In particular, if four lines of S have the property that five of the six pairs of lines are coplanar, then so is the sixth pair.

Proof. See [5].

(II) If $LM \circ LN$, and $y \sim L \vee M, L \vee N$, then $(y \vee L) \wedge (y \vee M) = (y \vee L) \wedge (y \vee N)$.

Proof. There is an $x \sim L \vee M, L \vee N$ with $(x \vee L) \wedge (x \vee M) = (x \vee L) \wedge (x \vee N) = X$ a line. Fix $p < L$. In the projective space underlying $\mathscr{L}^p$, the subspaces $L \vee M \vee x$ and $L \vee N \vee y$ have dimension 2, contain L, and hence span a subspace S of dimension at most 4 having L in its radical. Since $n \geqslant 3$, there is a point $p \vee z$ of $\mathscr{L}^p$ in $S^\perp$ on neither $L \vee M$ nor $L \vee N$. Let $Z = (z \vee L) \wedge (z \vee M)$. Then $Z < (L \vee M \vee x) \vee (p \vee z) \neq 1$ (since $L \vee M \vee x$ and $p \vee z$ are perpendicular in $\mathscr{L}^p$). By (I) applied to L, M, X, Z, we find that X and Z are coplanar. Then $Z = (z \vee L) \wedge (z \vee N)$ (by (I) applied to L, X, Z, N). Two further applications of (I) complete the proof.

(III) If $M_1M_2 \circ M_2M_3 \circ \cdots \circ M_rM_{r+1}$, $r \geqslant 3$, then there exists an M satisfying $M_1M_2 \circ M_1M \circ MM_{r+1}$ and $M \sim M_1 \vee M_2$.

Proof. Suppose $r = 3$. Then (ii) provides an $x \sim M_i \vee M_{i+1}$ for $i = 1, 2, 3$. By (II), $(x \vee M_i) \wedge (x \vee M_{i+1})$ is a line M independent of i. Then all requirements are met.

Now suppose $r > 3$. By induction, there is an N satisfying $M_2N \circ NM_{r+1}$. Now $M_1M_2 \circ M_2N \circ NM_{r+1}$, and we are back in the $r = 3$ case.

(IV) If $LM \equiv LN$ then $LM \circ LN$ or $M = N$.

Proof. By (III), there is a line X satisfying $ML \circ MX \circ XN$ and $X \sim L \vee M$. Use $x < X$ to establish $ML \circ LN$ if $M \neq N$.

(V) If $KL \equiv MN$, $K \wedge L = 0$, and $L, M > p$, then $L = M$.

Proof. By (III), there is an X with $KL \circ LX \circ XM$. Now use $x \sim K \vee L$, $L \vee X$, $X \vee M$ to complete the proof.

Proof of Lemma 1. Part (b) is just (V). Suppose $K \wedge L = p$ and $\alpha < M$. By (III), there exists N with $KL \circ LN \circ NM$. If $KL \circ LN$ via x then $(x \vee K) \wedge (x \vee L)$ must be $x \vee p$, from which $p < N$ follows. Similarly, $p < M$. Conversely, every line $M > p$ is coplanar with some line $N > p$ coplanar with L, and $KL \circ LN \circ NM$. This proves the lemma.

Definitions and Conventions. For α as in Lemma 1(a), we identify α with p. If $\alpha \neq p$ and there is a line $> \alpha, p$, this line is denoted $\alpha \vee p = p \vee \alpha$ and we write $p \sim \alpha$.

The *ideal line* $E \# F$ determined by distinct planes $E \sim F$ is defined by

$$E \# F = \{\alpha \mid \exists LM \in \alpha \text{ with } L < E, M < F\}.$$

We write $E \# F < E, F$ and $\alpha < E$, $\alpha < E \# F$ for α as in the definition. There is an obvious definition for *collinear* ideal points. According to (I), this is the "correct" definition inside $E \vee F$. As in the case of ideal points, we have to show that $E \# F$ can be equally well computed using other pairs of planes.

(VII) Suppose $E \sim F$, T is a 3-space $> E$, $p < T$, but $p \nless E$. Then there exists a plane D with $p < D < T$ and $E \# F = E \# D$.

Proof. Fix $x < E$. In $\mathscr{L}^x$, $E \vee F$ and T are planes on E. There is thus a 3-space $T' > E$ of $\mathscr{L}$ with $T' \sim E \vee F, T$. By (I), inside $T' \vee F$ there is a plane F' satisfying $F' < T'$ and $E \# F = E \# F'$. Similarly, there is a plane D with $p < D < T$ and $E \# F' = E \# D$. Similarly:

(VIII) If $\alpha < L < E$, then every point $p < E, p \nless L$, is on a line $N < E$ satisfying $LN \in \alpha$.

Proof. Let $LM \in \alpha$ and $x \sim E, L \vee M$. Set $X = (x \vee L) \wedge (x \vee M)$. Then $LX \in \alpha$. Now (I) applies (within any 4-space $> x \vee E = X \vee E$).

(IX) Ideal points α, β are collinear iff $\alpha < L$, $\beta < M$ for some coplanar lines L, M.

Proof. If $\alpha, \beta < E \# F$, then the desired lines can be found in E.

Conversely, assume $\alpha < L$, $\beta < M$, and $L \vee M = E$ is a plane. By (VIII), $LL' \in \alpha$ and $MM' \in \beta$ for some $L', M' < E$. Let $x \sim E$. Then $X = (x \vee L) \wedge (x \vee L')$ and $Y = (x \vee M) \wedge (x \vee M')$ are lines by (II), and both are on $x \vee E$. Thus, $F = X \vee Y$ is a plane, and $\alpha, \beta < E \# F$.

LEMMA 2. *Assume* (i) through (iii). *If* α *is an ideal point and* Λ *an ideal line, then* α *is collinear with some* $\beta < \Lambda$.

(X) For any p and α there is a point $q \sim p, \alpha$.

Proof. Let $K_0, K_1, \ldots, K_n$ be lines with $\alpha < K_0, p < K_n$, and each $K_i \wedge K_{i+1}$ a point. Assume $n \geqslant 3$, and consider K_0, K_1, K_2, K_3. There is a line $L_1 > K_0 \wedge K_1$ with $L_1 \sim K_0, K_1$. Now we have $K_2, K_1 \vee L_1 > K_1 \wedge K_2$, so there is a line L_2 coplanar with K_2 and satisfying $K_1 \wedge K_2 < L_2 < K_1 \vee L_1$.

By (ii), there is a point $x \sim K_0 \vee L_1, L_1 \vee L_2, L_2 \vee K_2$. In particular, $x \sim K_1 \wedge K_2$ and α by (IX). Now set $K_1' = x \vee \alpha$ and $K_2' = x \vee (K_1 \wedge K_2)$ and decrease n.

If $n = 2$, the same argument applies, this time with $x \sim p$ and α.

Proof of Lemma 2. We are given α and $\Lambda = E \# E'$. Pick $p < E$. By (*X*), there is a point $q \sim \alpha, p$. As above, there are planes F_1, F_2 such that $E \wedge F_1$ is a line, $p \vee q < F_1$, $q \vee \alpha < F_2$, and $F_1 \wedge F_2$ is a line. By (ii), there is an $x \sim E, F_1, F_2$. Then $x \sim \alpha$, and by (VII) there is a plane E'' with $E \# E' = E \# E''$ and $x < E'' < x \vee E$. Now $x \vee \alpha$ is coplanar with some line L satisfying $x < L < E''$. By (I), L and $E \# E''$ are on a common ideal point β. Since $\beta < E \# E''$ and $(x \vee \alpha) \vee (x \vee \beta)$ is a plane, this proves the lemma by (IX).

LEMMA 3. *Assume* (i) *through* (iv). *If* α *is collinear with two ideal points* β, γ *of an ideal line* Λ, *then* α *is collinear with every ideal point of* Λ.

Axiom (iv) is used as follows.

(XI) Suppose $\alpha < N_i$, and $L \wedge N_i = p_i$ is a point, for $i = 1, 2$. Then $N_1 \sim N_2$.

Proof. By (III), there is a line $N > \alpha$ coplanar with both N_1 and N_2. Let $x < N$. Then $x \vee p_1 \vee p_2$ is a plane by (iv), and $x \vee p_i$ is perpendicular to $N \vee N_{3-i}$ in $\mathscr{L}^x$. Thus, $x \vee p_2 \vee N_1$ is a 3-space T containing $N \vee p_1 \vee p_2$. It follows that $N, N_1, N_2 < T$, and hence that $N_1 \vee N_2$ is a plane by (I).

(XII) Let α, β, Λ be as in the lemma, with $\Lambda = E \# E'$. Then there is a point $x \sim \alpha, E$ with $x \vee \alpha \sim x \vee \beta$.

Proof. By (III) and (IX), there are lines $L_1 \sim L_2 \sim L_3 \sim L_4$ with $\beta < L_1 < E, \beta < L_3, \alpha < L_4$. Let $y \sim L_1 \vee L_2$, $L_2 \vee L_3$, $L_3 \vee L_4$. By (VIII), $y \sim \alpha$. By Lemma 1 and (VIII), $y \vee \beta$ exists and is coplanar with L_1. Since $y \sim L_3 \vee L_4$, we have $y \vee \alpha \sim y \vee \beta$. Any $x \sim E, L_1 \vee (y \vee \beta), (y \vee \beta) \vee (y \vee \alpha)$ meets our requirements.

Proof of Lemma 3. Let $x \sim \alpha, E$ with $x \vee \alpha \sim x \vee \beta$. By (VII), we may assume $x < E' < E \vee x$. Then let $y \sim \alpha, E'$ with $y \vee \alpha \sim y \vee \gamma$. We may assume $y < E < E' \vee y$. Then $y \vee \beta, y \vee \gamma < E$ by (VIII). We can apply (XI) to $L = x \vee y, N_1 = x \vee \alpha$, $N_2 = y \vee \alpha$ and obtain $x \vee \alpha \sim y \vee \alpha$. Then $x \vee y \sim x \vee \alpha$, so $(x \vee y) \vee (x \vee \alpha) \vee (x \vee \beta)$ is a 3-space containing $y \vee \alpha$ and $y \vee \beta$. Thus, $y \vee \alpha \sim y \vee \beta$. We already know $y \vee \alpha \sim y \vee \gamma$. Thus, using $\mathscr{L}^y$ we see $y \vee \alpha \sim E$. Then $y \vee \alpha \sim y \vee \lambda$ for all $\lambda < E \# E' = \Lambda$, as required (see (IX)).

Proof of the Theorem. Let α be an ideal point, $\alpha < L$, and $\alpha \neq p < L$. Pick any $q \sim p$ not coplanar with L. Then q and α cannot be collinear by (XI). In view of Lemmas 2 and 3, the Buekenhout–Shult theorem [4] now completes the proof.

Examples. A lattice associated with the simple group F_{22} (see Buekenhout and Hubaut [3]) has $n = 2$ in (i): $\mathscr{L}^x$ is of type $SU(6, 2)$. In this case, (I) even fails: The points and planes in a 3-space form a Steiner system $S(22, 6, 1)$. However, for the case of $SU(7, K)$, $O(7, K)$ and $O^-(8, K)$, assuming (ii) through (iv) the theorem should still be true.

That (i), (iii), and (iv) do not imply (ii) is seen from the following examples. Let V be an orthogonal or unitary vector space, and let $\mathscr{L}$ consist of $\varnothing$, V, the vectors in V, and the translates of all totally singular subspaces. Of course, one can also delete some such points and subspaces and still arrange to have a locally polar lattice. If instead V had been chosen to be symplectic, then (ii) and (iv) would both fail.

Examples satisfying (ii), (iii), usually (i), but not (iv), are constructed as follows. Let K be a field of characteristic 2, V a nondegenerate orthogonal vector space over K, and R a nonsingular 1-space. Then $\mathscr{L}$ consists of ($\varnothing$, 1, and) all $(X + R)/R = \bar{X}$ for X a totally singular subspace not contained in $R^\perp$. If $W \nleqslant R^\perp$ is a singular 1-space, then $\mathscr{L}^{\bar{W}}$ is clearly just the polar space for $W^\perp/W$. Thus, $\mathscr{L}$ is locally polar. If K is perfect, an easy computation shows that any two points are collinear! Since any K contains $GF(2)$, it follows that (iv) fails. Note that, if $K \neq GF(2)$, then one can again delete some elements of $\mathscr{L}$ and still arrange to have a locally polar lattice. Lemmas 1 and 2 hold here, the ideal points being those of $\mathscr{L}$ along with all 1-spaces of $R^\perp/R$.

REFERENCES

1. F. BUEKENHOUT, Extensions of polar spaces and the doubly transitive symplectic groups, to appear.
2. F. BUEKENHOUT, Diagrams for geometries and groups, to appear.
3. F. BUEKENHOUT AND X. HUBAUT, Locally polar spaces and related rank 3 groups, *J. Algebra* **45** (1977), 391–434.
4. F. BUEKENHOUT AND E. SHULT, On the foundations of polar geometry. *Geom. Ded.* **3** (1974), 155–170.
5. W. M. KANTOR, Dimension and embedding theorems for geometric lattices. *J. Combinatorial Theory A* **17** (1974), 173–195.
6. J. TITS, "Buildings of Spherical Type and Finite BN-Pairs," Lecture Notes in Mathematics No. 386, Springer-Verlag, Berlin, 1974.

Printed in Belgium